Grading guide, Pricing Financial Assets, June 2021

1. Consider a stock with price $\mathbf{S}_t$ at time t and a zero coupon bond maturing at time $T > t$ with price $\mathbf{P}(t,T)$ at time t.

   (a) Assume that the stock pays no dividend. What is the forward price $F(t,T)$ at t for the stock with delivery at T?

   (b) Now assume that the stock pays a continuous dividend rate of δ. What is the forward price in this case?

   (c) Let $c(t,K,T)$ and $p(t,K,T)$ denote the prices respectively of a call and a put option on the stock, both with exercise price K and maturity T. What is the put-call-parity for the case with the dividend paying stock?

   (d) If the prices of the call, put and stock as defined above are known at t, derive the implied dividend rate.

Solution:

(a) Consider the following two strategies to get 1 stock at time T:

(a) Enter into a forward contract with forward price $F(t,T)$ to get the stock at T and at the same time buy $F(t,T)$ zero coupon bonds. The total payment at t is $\mathbf{P}(t,T)\,F(t,T)$. At time T the zero coupon bond portfolio matures at gives the $F(t,T)$ to be paid to receive the stock.

(b) Buy the stock at time t and keep it to time T.

If there is to be no arbitrage opportunities the to different outlays at time t must be identical, i.e.

$$\mathbf{S}(t) = \mathbf{P}(t,T)\,F(t,T)$$
$$F(t,T) = \mathbf{S}(t)/\mathbf{P}(t,T)$$

(b) If there is a continuous dividend rate of δ on the stock, in strategy (b) buy only $\mathsf{e}^{-\delta(T-t)}$ of the stock, but reinvest dividends continuously to get 1 stock at time T. Thus

$$\mathsf{e}^{-\delta(T-t)}\mathbf{S}(t) = \mathbf{P}(t,T)\,F(t,T)$$

or

$$F(t,T) = \mathsf{e}^{-\delta(T-t)}\mathbf{S}(t)/\mathbf{P}(t,T)$$

(c) A long call and a short put with same strike K and maturity T is equivalent to at long forward with delivery price K and maturity T. The value of this forward contract is $\mathbf{P}(t,T)\,(F(t,T)-K)$. So (abbreviating the notation a little)

$$\mathbf{P}(t,T)\,(F(t,T)-K) = c_t - p_t$$

or

$$\mathsf{e}^{-\delta(T-t)}\mathbf{S}(t) - \mathbf{P}(t,T)\,K = c_t - p_t$$

(d) Solve

$$\mathrm{e}^{-\delta(T-t)}\mathbf{S}(t) = c_t - p_t + \mathbf{P}(t,T)\,K$$

or

$$\delta = -\tfrac{1}{T-t}\ln\frac{c_t - p_t + \mathbf{P}(t,T)\,K}{\mathbf{S}(t)}$$

(Hull 9th ed, section 17.3-17.4).

2. Consider a situation with credit risk, and let the probability of a borrower not defaulting at or before time t be given by $V(t)$, $V(0) = 1$.
   (a) What is the probability that the borrower will default between time t and $t + \Delta t$ conditional on not being in default at time t? Use this to define the continuously compounded default hazard rate λ.
   (b) How, and under which assumptions, may we estimate the hazard rate from the interest rate spread s on bonds issued by the borrower? Under what probability measure would we say this estimate is derived? Compare this to a hazard rate that is derived from default frequencies and recovery ratios published by a rating agency.
   (c) In one model by Merton the value of a claim on a company with limited liability is modelled using a variation of the Black-Scholes-Merton option model. What is the option features embedded in such a claim? What parameters that are not directly observable must be determined to price the claim in this model? Comment on the model.

Solution:

(a) The conditional default probability from t to $t + \Delta t$ is

$$\frac{V(t) - V(t+\Delta t)}{V(t)}$$

Given $V(t)$ the hazard rate is the rate of decay of survivors. Assuming differentiability you may define it as

$$\lambda(t) = -\frac{\frac{\partial V(t)}{\partial t}}{V(t)}$$

This can be motivated by considering the above discrete time step and letting the time step approach zero (Hull 9th ed, section 24.2-24.4).

(b) Under a risk neutral probability measure the hazard rate may be derived from prices of traded assets, e.g. bonds from the company in question. Assuming that the recovery rate Rec is known you may estimate the average hazard rate as the hazard rate that makes the discounted expected payoff (i.e. taking defaults into account) to be the price of a risk free bond. Assuming the risky bond pays a spread of s over the risk free bond for a given maturity T you can estimate the average hazard rate over the full period by

$$\bar{\lambda}(T) = \frac{s(T)}{1 - Rec}$$

You may then use this to bootstrap a hazard rate structure from a term structure of credit spreads.

This analysis should, however, be seen as conducted under a risk neutral ("$\mathbb{Q}$") measure, so that (λ, Rec) will not be the same as frequencies and averages published by rating agencies, real world probabilities ("$\mathbb{P}$") (Hull 9th ed., section 24.5).

(c) In the Merton-model the value of risky debt issued by a limited liability entity is modelled as the value of the assets of the entity less the value of the equity held by the owners of the company. Due to the limited liability the owners can be seen to hold a call option on the assets.

Assuming there is only one form of debt, a zero coupon debt of face value D maturing at time T, the Black-Scholes-Merton Model can be applied. The underlying asset for that model would be the value of the entity assets V_t, which is typically unknown, as is it's volatility, that is assumed constant σ_V.

Note first that the value of equity at time T is

$$E_T = \max(V_T - D, 0)$$

where you can use the BSM-call price formula if you know asset values and their volatility.

Secondly, when both assets value and equity value are described by geometric Brownian motions you can use Ito's lemma to derive the relationship between the equity volatility σ_E and the asset volatility.

$$\sigma_E E_t = \frac{\partial E}{\partial V} \sigma_V V_t$$

Assuming that the value of the equity and the equity volatility is known from markets, you have two equations in two unknowns (V_t, σ_V) hat can be solved numerically for the value of the debt, using $D_t = V_t - E_t$.

A problem with Merton's model is that values are assumed to follow Ito-processes, i.e. with continuous sample paths. This makes the likelihood of a default a short time step ahead very small (no "jump-to-default"') and credit spreads to go to zero as the maturity of risky bonds goes to zero, which is not in line with empirical evidence (which may be due e.g. to asset values that fundamentally jump in case of a default or to asymmetric information between owners and creditors on the value of assets, revealed at default), (Hull 9th ed, section 24.6).

3. (a) Give a definition of a payer and a receiver interest rate swaption (also called a swap option)

(b) Give an argument that a receiver swaption can be seen as a call option on a properly defined fixed rate bond with no credit risk. What can you say about the coupon and principal of such a bond?

(c) Let PS denote the value of a swaption to pay a fixed rate s_K and receive LIBOR between times T_1 and T_2, let RS denote the value of a swaption to receive a fixed rate of s_K and pay LIBOR between times T_1 and T_2, and let RFS denote the value of a forward starting swap that receives a fixed rate of s_K and pays LIBOR between times T_1 and T_2.

Assume that there are no arbitrage opportunities and i) show that $PS + RFS = RS$ and ii) deduce that $PS = RS$ when s_K equals the current forward swap rate.

Solution:

(a) Cf. Hull p.659f (8th ed) and p.684f (9th ed)
Definition 0.1 (Swaption)**.** *A payer (receiver) swaption or swap option gives the holder the right to enter into a payer (receiver) Interest Rate Swap in the future*

(b) Cf. Hull p. 660 (8th ed), p 685 (9th ed) If the fixed rate coupon on the bond is equal to the swap rate the principal is the par value of the bond (barring credit risks, liquidity considerations etc.). The receiver swaption can be seen as an option to receive this bond by delivery of the principal (which we , somewhat idealized, assume will be equal to the continuing value of the variable leg of the swap). We can note that both swaption and bond option will increase in value with lower longer rates.

(c) Consider the payments from T_1 to T_2 in the following two cases that depend on the realized level of the swap rate s_{T_1} at T_1 with maturity at T_2:

Case $s_K < s_{T_1}$	Payment
PS	Pays s_K and receives $LIBOR$
RFS	Receives s_K and pays $LIBOR$
RS	0

Case $s_K \geq s_{T_1}$	Payment
PS	0
RFS	Receives s_K and pays $LIBOR$
RS	Receives s_K and pays $LIBOR$

From the table it is seen that the payments for the payer swaption and the forward starting swap equals the payments on the receiver swaption. Thus barring arbitrage the equation must hold. If s_K is the current forward swap rate the value RFS is by definition zero.

(Hull 9th ed., practice question 29.18).